Gravitomagnetism in the Kerr-Newman-Taub-NUT spacetime

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Abstract. We study the motion of test particles and electromagnetic waves in the Kerr-Newman-Taub-NUT spacetime in order to elucidate some of the effects associated with the gravitomagnetic monopole moment of the source. In particular, we determine in the linear approximation the contribution of this monopole to the gravitational time delay and the rotation of the plane of the polarization of electromagnetic waves. Moreover, we consider "spherical" orbits of uncharged test particles in the Kerr-Taub-NUT spacetime and discuss the modification of the Wilkins orbits due to the presence of the gravitomagnetic monopole.

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1. Introduction

In its simplest manifestation, gravitomagnetism is caused by the current of mass-energy in direct analogy with electrodynamics. In particular, gravitomagnetic effects can be found in the external gravitational field of rotating astronomical masses.

In this article we are interested in the exterior Kerr-Newman-Taub-NUT (KNTN) solution given in the Boyer-Lindquist coordinates by the metric [1, 2, 3, 4, 5]

$$\begin{aligned} \mathrm{d}s^2 = & -\frac{1}{\Sigma}(\Delta - a^2\sin^2\theta)\mathrm{d}t^2 + \frac{2}{\Sigma}[\Delta\chi - a(\Sigma + a\chi)\sin^2\theta]\mathrm{d}t\mathrm{d}\phi \\ & + \frac{1}{\Sigma}[(\Sigma + a\chi)^2\sin^2\theta - \chi^2\Delta]\mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma d\theta^2 \end{aligned} \qquad (1.1)$$

and the corresponding electromagnetic Faraday tensor can be expressed in terms of the 2-form

$$\begin{aligned} F = \frac{Q}{\Sigma^2}\{[r^2 - (\ell + a\cos\theta)^2]\mathrm{d}r \wedge (\mathrm{d}t - \chi\mathrm{d}\phi) \\ + 2r\sin\theta(\ell + a\cos\theta)\mathrm{d}\theta \wedge [(r^2 + a^2 + \ell^2)\mathrm{d}\phi - a\mathrm{d}t]\}. \end{aligned} \tag{1.2}$$

Here Σ, Δ, and χ are defined by

$$\Sigma = r^2 + (\ell + a\cos\theta)^2, \ \Delta = r^2 - 2Mr - \ell^2 + a^2 + Q^2, \ \chi = a\sin^2\theta - 2\ell\cos\theta \ . \tag{1.3}$$

Units are chosen such that $G = c = 1$, so that (M, Q, a, ℓ) all have the dimension of length: the source has mass M, electric charge Q, angular momentum $J = Ma$ (i.e., gravitomagnetic dipole moment) along the z-direction, and gravitomagnetic monopole moment $\mu = -\ell$, where ℓ is the NUT parameter. In this family the charged KNTN metric can be obtained from the uncharged Kerr-Taub-NUT (KTN) metric by the simple transformation $M \to M - Q^2/(2r)$ in Δ, while the Taub-NUT metric corresponds to $a = 0$ [6, 7, 8, 9]. It follows from equation (1.2) that in the far zone ($r \gg a$ and $r \gg |\ell|$), the lowest-order contribution to the electric field is the radial Coulomb field, while the lowest-order contribution to the magnetic field is due to the field generated by the magnetic dipole moment Qa of the source as well as terms of similar order proportional to $2Q\ell$.

An interesting example of a gravitomagnetic phenomenon is provided by the clock effect for circular equatorial orbits around rotating masses [10, 11]. To illustrate the situation in a more general context, consider the circular equatorial motion of a test particle of mass m and negative electric charge $-q < 0$ around the Kerr-Newman black hole of mass M, charge $Q > 0$, and angular momentum $J = Ma$. Here we neglect all (electromagnetic and gravitational) radiative effects for the sake of simplicity. A Newtonian analysis implies that the test particle moves along a circle of radius r with uniform angular frequency given by $(\frac{M+\eta Q}{r^3})^{1/2}$, where $-\eta = -q/m < 0$ is the charge-to-mass ratio of the test mass. This Newtonian result is expected to emerge from the following general relativistic analysis when the test particle is very far from the black hole. In the relativistic theory, the Lorentz equation of motion

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = -\eta F^\mu{}_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \tag{1.4}$$

must be solved. Here τ is the proper time of the test particle along its world line. A detailed analysis reveals that in this case Eq. (1.4) reduces to

$$\omega_K^2 \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} - a\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 - \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 = \omega_C^2 \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} - a\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right) \ , \tag{1.5}$$

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(\frac{2M}{r} - \frac{Q^2}{r^2}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau} - a\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 - (r^2 + a^2)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 = 1 \ , \tag{1.6}$$

where $\pm\omega_K$ ($\omega_K > 0$) defined by

$$\omega_K^2 = \frac{M}{r^3} - \frac{Q^2}{r^4} \tag{1.7}$$

is the Keplerian frequency for a neutral test particle in circular orbit around a Reissner-Nordström black hole and $\pm\omega_C$ ($\omega_C \geq 0$) defined by

$$\omega_C^2 = \eta\frac{Q}{r^3} \tag{1.8}$$

would be the frequency of motion of the test particle of charge $-\eta m$ if only the Coulomb interaction were present.

To analyze the system (1.5)–(1.6), first set $a = 0$; then $(d\phi/dt)^2$ satisfies a quadratic equation that has two possible solutions. These can be distinguished by the fact that in one case $dt/d\tau > 0$ and in the other case $dt/d\tau < 0$. We must impose the physical requirement that the time coordinate t increases along timelike world lines. With this postulate, one then obtains $\mathrm{d}\phi/\mathrm{d}t = \pm\Omega$, where $\Omega > 0$ is given by

$$\Omega^2 = \omega_K^2 - \frac{1}{2} r^2 \omega_C^4 + N\omega_C^2 \tag{1.9}$$

with

$$N = \sqrt{1 - \frac{3M}{r} + \frac{2Q^2}{r^2}(1 + \frac{\eta^2}{8})}\ . \tag{1.10}$$

Here Ω^2—in contrast to the second solution that we have ruled out—properly reduces to the Newtonian result far from the black hole.

Next consider a charged rotating black hole ($a \neq 0, Q \neq 0$). The results can be simply stated for $a/M \ll 1$, in which case to first order in a/M one finds

$$\frac{\mathrm{d}t}{\mathrm{d}\phi} = \pm\frac{1}{\Omega} + a(1 - \mathcal{Q})\ , \tag{1.11}$$

where

$$\mathcal{Q} = \frac{1}{2} \frac{\eta Q}{N[M - \frac{Q^2}{r}(1 + \frac{1}{2}\eta^2) + \eta Q N]}\ . \tag{1.12}$$

Thus if t_+ (t_-) denotes the coordinate time period for a complete revolution around the black hole in the counterclockwise (clockwise) sense ($\Delta\phi = \pm 2\pi$), then

$$t_+ - t_- = 4\pi a(1 - \mathcal{Q})\ , \tag{1.13}$$

which is the expression for the single-clock clock effect [10, 11] in this case. Note that as $r \to \infty$ this result reduces to

$$t_+ - t_- = 4\pi a \frac{M + \frac{1}{2}\eta Q}{M + \eta Q}\ , \tag{1.14}$$

so that the effect is always less than $4\pi a$, the result for a neutral particle. It is interesting to investigate the two-clock clock effect [10, 11] in this case as well, i.e. $\tau_+ - \tau_-$, where τ is the corresponding proper time as measured along the geodesics. One finds

$$\frac{\mathrm{d}\tau}{\mathrm{d}\phi} = \pm\frac{\Omega^2 - \omega_K^2}{\Omega\omega_C^2} + a(1 - \mathcal{Q}')\ , \tag{1.15}$$

where

$$1 - \mathcal{Q}' = \frac{1}{2} N^{-1}(1 + \frac{\omega_K^2}{\Omega^2})\ . \tag{1.16}$$

It follows that

$$\tau_+ - \tau_- = 4\pi a(1 - \mathcal{Q}') \tag{1.17}$$

and that $\mathcal{Q}' \to \mathcal{Q}$ as $r \to \infty$. These results, valid only to first order in a/M, indicate a certain gravitomagnetic temporal structure around the Kerr-Newman black hole that is consistent with the axial symmetry of the underlying spacetime structure. It is the purpose of this paper to clarify how gravitomagnetism is affected by the presence of the gravitomagnetic *monopole* moment $\mu = -\ell$, which has *spherical* symmetry.

In the following section, the gravitomagnetic field is studied and the results are applied in section 3 to the propagation of test electromagnetic waves in the Kerr-Taub-NUT spacetime. Section 4 describes circular holonomy in this spacetime. Spherical orbits are considered in section 5. This is followed in the final section by a discussion of our results.

2. Gravitomagnetic field

Even though the spacetime curvature vanishes asymptotically at large spatial distances, the KNTN spacetime is not asymptotically flat when $\ell \neq 0$, a fact intimately related to the gravitomagnetic monopole character of the source associated with this parameter. At present there is no evidence for the existence of either magnetic or gravitomagnetic monopoles. Moreover, the lack of asymptotic flatness for a spacetime describing a localized source appears to be unphysical. However, like the Gödel spacetime, the Kerr-Newman-Taub-NUT spacetime can play a useful role in clarifying important gravitomagnetic features of classical general relativity.

With this goal in mind, we set $Q = 0$ and expand the metric to first order in a and $\mu = -\ell$; in this way one can more easily interpret both parameters in the context of gravitomagnetism. One finds

$$\begin{aligned} \mathrm{d}s^2 = & -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2) \\ & - 4\left[\frac{aM}{r}\sin^2\theta - \mu\cos\theta\left(1-\frac{2M}{r}\right)\right]\mathrm{d}t\mathrm{d}\phi\ , \end{aligned} \tag{2.1}$$

namely, the Schwarzschild metric plus the Lense-Thirring and Taub-NUT terms. While the time-coordinate slices are intrinsically asymptotically flat, the fact that $g_{t\phi} \to -2\ell\cos\theta$ as $r \to \infty$ implies that the spacetime is not asymptotically flat [6, 7].

By introducing the isotropic radial coordinate ρ defined by $r = \rho(1+\Phi/2)^2$, where $\Phi = M/\rho$ is the (sign-reversed) Newtonian gravitational potential, and the corresponding isotropic Cartesian-like coordinates $x = \rho\sin\theta\cos\phi$, $y = \rho\sin\theta\sin\phi$, and $z = \rho\cos\theta$ (in 3-vector form $\boldsymbol{\rho} = \rho\hat{\boldsymbol{\rho}} = (x, y, z)$), the metric takes the form

$$\begin{aligned} \mathrm{d}s^2 = & -\left(\frac{1-\Phi/2}{1+\Phi/2}\right)^2\mathrm{d}t^2 + (1+\Phi/2)^4\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right) \\ & - 4\mathbf{A}_{(\mathrm{g})}\cdot\mathrm{d}\boldsymbol{\rho}\,\mathrm{d}t\ . \end{aligned} \tag{2.2}$$

We interpret this metric as representing a background Minkowski spacetime in inertial coordinates (t, x, y, z) together with a source characterized by a gravitoelectric potential Φ and a linearized gravitomagnetic vector potential $\mathbf{A}_{(\mathrm{g})}$ given by

$$\mathbf{A}_{(\mathrm{g})} = \mathcal{F}(\hat{\mathbf{J}}\times\boldsymbol{\rho}) \leftrightarrow \mathbf{A}_{(\mathrm{g})}\cdot\mathrm{d}\boldsymbol{\rho} = \mathcal{F}\rho^2\sin^2\theta\ \mathrm{d}\phi \tag{2.3}$$

(i.e., the slicing gravitomagnetic potential 1-form $N_a\mathrm{d}x^a = g_{ta}\mathrm{d}x^a$ when multiplied by -2, see [12]). Here $\mathbf{J} = Ma\,\hat{\mathbf{z}}$ is the angular momentum,

$$\mathcal{F} = \frac{aM}{\rho^3(1+\Phi/2)^2} - \frac{\mu z}{\rho(x^2+y^2)}\left(\frac{1-\Phi/2}{1+\Phi/2}\right)^2\ , \tag{2.4}$$

and the 3-vector operations are carried out in the flat spatial metric, so that

$$(\hat{\mathbf{z}}\times\boldsymbol{\rho})\cdot\mathrm{d}\boldsymbol{\rho} = \rho^2\sin^2\theta\ \mathrm{d}\phi = (\rho^2-z^2)\mathrm{d}\phi\ . \tag{2.5}$$

In the case that $M/\rho \ll 1$ far from the source, one has

$$\mathcal{F} \approx \frac{Ma}{\rho^3} - \frac{\mu z}{\rho(x^2+y^2)} \ , \ \mathbf{A}_{(\mathrm{g})} \approx \frac{\mathbf{J}\times\boldsymbol{\rho}}{\rho^3} - \frac{\mu z\,\hat{\mathbf{z}}\times\boldsymbol{\rho}}{\rho(x^2+y^2)} \ , \tag{2.6}$$

and the gravitomagnetic vector field $\mathbf{B}_{(\mathrm{g})} = \boldsymbol{\nabla}\times\mathbf{A}_{(\mathrm{g})}$ is given by

$$\mathbf{B}_{(\mathrm{g})} = \frac{J}{\rho^3}[3(\hat{\boldsymbol{\rho}}\cdot\hat{\mathbf{J}})\hat{\boldsymbol{\rho}} - \hat{\mathbf{J}}] + \frac{\mu\boldsymbol{\rho}}{\rho^3} \ . \tag{2.7}$$

Thus J corresponds to the gravitomagnetic dipole moment and μ to the gravitomagnetic monopole moment. The threading gravitomagnetic vector potential (i.e., the 1-form $M_a \mathrm{d}x^a = -(g_{ta}/g_{tt})\mathrm{d}x^a$, see [12]) defines another vector

$$\boldsymbol{\gamma} = -2\mathcal{F}\left(\frac{1+\Phi/2}{1-\Phi/2}\right)^2 \hat{\mathbf{J}}\times\boldsymbol{\rho} = -2\frac{\mathbf{J}\times\boldsymbol{\rho}}{\rho(\rho-M/2)^2} + 2\frac{\mu z\,\hat{\mathbf{z}}\times\boldsymbol{\rho}}{\rho(x^2+y^2)} \ , \tag{2.8}$$

so that $\boldsymbol{\gamma} \approx -2\mathbf{A}_{(\mathrm{g})}$ for $\rho \gg M$.

One can now use the metric (2.2) to treat phenomena in terms of a gravitoelectric field $\mathbf{E}_{(\mathrm{g})} = -\boldsymbol{\nabla}\Phi$ and a gravitomagnetic field $\mathbf{B}_{(\mathrm{g})} = -\boldsymbol{\nabla}\Psi$, where

$$\Psi = \frac{\mathbf{J}\cdot\boldsymbol{\rho}}{\rho^3} + \frac{\mu}{\rho} \tag{2.9}$$

is the gravitomagnetic scalar potential [13]. In the definition of gravitoelectric and gravitomagnetic fields, certain conventional numerical factors of 2, etc., are unavoidable depending on the specifics of the analogy with electromagnetism. It turns out that in the description of the linearized gravitational motion of a test particle of inertial mass m around a central source of inertial mass M, one can employ the *same* formulas as in classical electrodynamics if one assigns a gravitoelectric charge M and a gravitomagnetic charge $2M$ to the central source and corresponding gravitational charges of $-m$ and $-2m$ to the test mass [13]. The different signs ensure that gravitation is attractive; moreover, the ratio of gravitomagnetic charge to the gravitoelectric charge is always 2 due to the spin-2 character of the linearized gravitational field. We employ this convention in the present work.

Geodesic motion of test particles in the field of a gravitomagnetic monopole has been considered in [14, 15] and in references cited there, while lensing aspects of a gravitomagnetic monopole have been discussed briefly in [15]. In the following section we consider the effect of this monopole on the propagation of electromagnetic waves.

3. Wave propagation

Consider the propagation of electromagnetic waves in the Kerr-Taub-NUT spacetime. We work to linear order in a and μ and express the electromagnetic perturbations in terms of the background Cartesian-like coordinates (t, x, y, z) introduced above. Writing the sourcefree Maxwell equations in these coordinates as $F_{[\alpha\beta,\gamma]} = 0$ and $[\sqrt{-g}F^{\alpha\beta}]_{,\beta} = 0$ and introducing the standard decompositions $F_{\alpha\beta} \to (\mathbf{E}, \mathbf{B})$ and $\sqrt{-g}F^{\alpha\beta} \to (-\mathbf{D}, \mathbf{H})$ [16], one finds the equivalent flat spacetime Maxwell equations in inertial coordinates in the presence of an effective material medium [16, 17], i.e.,

$$\nabla\cdot\mathbf{D} = 0 \ , \ \nabla\cdot\mathbf{B} = 0 \ , \ \nabla\times\mathbf{E} = -\partial_t\mathbf{B} \ , \ \nabla\times\mathbf{H} = \partial_t\mathbf{D} \ . \tag{3.1}$$

Therefore we may think of the electromagnetic field as propagating in an inertial frame in Minkowski spacetime but in the presence of a gravitational "medium" with certain constituitive properties given by [16, 17, 18, 19]

$$D_i = \epsilon_{ij}E_j - [\boldsymbol{\gamma}\times\mathbf{H}]_i \ , \ B_i = \mu_{ij}H_j + [\boldsymbol{\gamma}\times\mathbf{E}]_i \ , \tag{3.2}$$

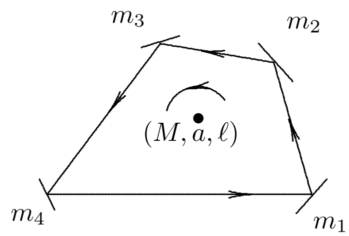

Figure 1. Propagation of null rays along a closed contour around a mass M with gravitomagnetic monopole ($\mu = -\ell$) and dipole ($J = Ma$) moments. The "mirrors" m_1–m_4 in this schematic diagram could be transponders on-board artificial satellites.

where $\epsilon_{ij} = \mu_{ij} = -\sqrt{-g}g^{ij}/g_{tt}$ and $\gamma_i = -g_{ti}/g_{tt}$ ($= M_i$, the threading shift 1-form as above). For the present metric one has

$$\epsilon_{ij} = \mu_{ij} = \mathcal{N}\delta_{ij} \ , \ \mathcal{N} = \frac{(1+\Phi/2)^3}{1-\Phi/2} \ , \tag{3.3}$$

and $\boldsymbol{\gamma}$ is given by Eq. (2.8). Here $\mathcal{N}$ can be interpreted as an effective index of refraction of the gravitational medium and $\boldsymbol{\gamma}$ represents the gyrotropic aspects of the medium. We note that $\boldsymbol{\gamma}$ is singular along the entire z-axis due to the presence of the term involving $\mu \neq 0$. However, this singularity can be removed by a local coordinate transformation in any given exterior neighborhood [1,6], a fact that is consistent with the gauge dependence of the vector potential.

Since these constitutive quantities are all independent of time, the electromagnetic waves propagate with constant frequency ω in this gravitational medium. With a time dependence of the form $\exp(-i\omega t)$, Maxwell's equations then reduce to the wave equation

$$(i^{-1}\boldsymbol{\nabla} - \omega\boldsymbol{\gamma}) \times \mathbf{W} = -i\omega\mathcal{N}\mathbf{W} \tag{3.4}$$

for the complex Kramers vector $\mathbf{W} = \mathbf{E} + i\mathbf{H}$. Note that the gravitomagnetic contribution to this Dirac-type equation enters via the substitution $i^{-1}\boldsymbol{\nabla} \to i^{-1}\boldsymbol{\nabla} - \omega\boldsymbol{\gamma}$. Therefore, it follows from the Aharonov-Bohm effect that radiation following a closed spatial path C—via reflection from devices such as mirrors, for example (see Fig. 1)—in the exterior field of a gravitomagnetic source exhibits the following phase difference comparing the two waves after one loop in the counterclockwise (+) direction and one loop in the clockwise (−) direction along the closed path C [18]

$$\varphi_+ - \varphi_- = 2\omega \oint_C \boldsymbol{\gamma} \cdot \mathrm{d}\boldsymbol{\rho} \ . \tag{3.5}$$

For $\rho \gg M$, $\boldsymbol{\nabla} \times \boldsymbol{\gamma} \simeq -2\mathbf{B}_{(\mathrm{g})}$; therefore, one finds that $\varphi_+ - \varphi_-$ is -4ω times the gravitomagnetic flux threaded by the closed path C.

If we consider matter waves propagating around this path, corresponding to radiation of particles with nonzero rest mass $m > 0$ instead of photons, then ω in Eq. (3.5) must be interpreted as the de Broglie frequency of the matter waves, i.e. $\omega \to m/\hbar$. Moreover, a different treatment is required if instead of C a path defined by optical fibers (or a ring laser) is employed. Clearly the constitutive properties of the medium should then be taken into account as well.

If the dominant wavelength of the waves is negligible compared to the characteristic length scales of the medium, one can interpret Eq. (3.5) in terms of

rays of radiation. In this case, let P be an observer at a point on the closed path C. Then according to this observer the interval of the coordinate time t_+ (t_-) for the rays to go around C in the positive (negative) sense is such that $\varphi_+ - \varphi_- = \omega(t_+ - t_-)$. Thus

$$t_+ - t_- = 2\oint_C \boldsymbol{\gamma}\cdot \mathrm{d}\boldsymbol{\rho}\ , \tag{3.6}$$

which is reminiscent of the Sagnac effect via the gravitational Larmor theorem [13].

Consider now a closed path C in the (x, y)-plane far from the source ($\rho \gg M$) and note that the monopole contribution to $t_+ - t_-$ in Eq. (3.6) vanishes due to the fact that $z = 0$ in this plane (see Eq. (2.8)). It should also vanish in any other plane through the origin by the spherical symmetry of the monopole. This is clear from a simple application of Stokes' theorem (see below) if C does not encircle the singularity at $\rho = 0$. If it does, then let C_0 be a circle in this plane around the origin, such that C is always outside C_0. It follows from Stokes' theorem that

$$\oint_{C-C_0} \frac{z[\hat{\mathbf{z}}\times\hat{\boldsymbol{\rho}}]\cdot \mathrm{d}\boldsymbol{\rho}}{x^2+y^2} = -\int \frac{\boldsymbol{\rho}\cdot \mathrm{d}\mathbf{S}}{\rho^3} = 0 \tag{3.7}$$

since $\boldsymbol{\rho}$ in this plane is orthogonal to the area element $\mathrm{d}\mathbf{S}$ by definition. It now remains to calculate

$$I = \oint_{C_0} \frac{z[\hat{\mathbf{z}}\times\hat{\boldsymbol{\rho}}]\cdot \mathrm{d}\boldsymbol{\rho}}{x^2+y^2} \tag{3.8}$$

in this plane and to show that it vanishes regardless of the radius r_0 of the circle C_0. Taking advantage of the axial symmetry of the "medium", it suffices to calculate I in the (x', y')-plane, where the (x', y', z') coordinate system is obtained from the (x, y, z) system by a rotation of angle α about the x-axis. Thus

$$x = x', \quad y = y'\cos\alpha - z'\sin\alpha, \quad z = y'\sin\alpha + z'\cos\alpha. \tag{3.9}$$

A straightforward calculation then results in

$$I = \sin\alpha\cos\alpha \int_0^{2\pi} \frac{\sin\phi\ \mathrm{d}\phi}{\cos^2\alpha + \sin^2\alpha\cos^2\phi} = 0\ , \tag{3.10}$$

where $x' = r_0\cos\phi$, $y' = r_0\sin\phi$, and $z' = 0$ along C_0.

It is interesting to point out that an equation very similar to Eq. (3.4) would hold in a background with Cartesian-like coordinates for the propagation of electromagnetic waves in the general Kerr-Taub-NUT spacetime. In this spacetime, one must identify t with $t + n(8\pi\ell)$ for any integer n; that is, t is assumed to be periodic with period $8\pi\ell$ [1, 6]. Thus in the presence of a gravitomagnetic monopole moment a temporal delay equation of the form (3.6) should be interpreted modulo an integer multiple of $8\pi\ell$; moreover, a time dependence of the form $\exp(-i\omega t)$ implies that $4\omega\ell$ must be an integer. Hence the photon energy is quantized in units of $\hbar/(4\ell)$. Various aspects of this quantization condition have been explored in [20, 21]. We are interested in very high frequency radiation; therefore, this discreteness will not be pursued further here.

Let us note that in the JWKB limit, one can solve the wave equation (3.4) with the ansatz

$$\mathbf{W} = \mathbf{W}_0\, e^{-i\omega(t-t_0)+i\omega S(\mathbf{x})}, \tag{3.11}$$

where $\mathbf{W}_0 = \mathbf{W}(t_0, \mathbf{x}_0)$ and

$$S(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \boldsymbol{\gamma}(\mathbf{x}')\cdot \mathrm{d}\mathbf{x}' + \int_{\mathbf{x}_0}^{\mathbf{x}} \mathcal{N}(\mathbf{x}')\,\hat{\mathbf{k}}\cdot \mathrm{d}\mathbf{x}'. \tag{3.12}$$

Here $\hat{\mathbf{k}}$ is a constant unit vector that indicates the direction of wave propagation in the background inertial frame. It is useful to write the eikonal S in the form

$$S(\mathbf{x}) = \hat{\mathbf{k}} \cdot (\mathbf{x} - \mathbf{x}_0) + \int_{\mathbf{x}_0}^{\mathbf{x}} [\mathcal{N}(\mathbf{x}') - 1]\, \hat{\mathbf{k}} \cdot \mathrm{d}\mathbf{x}' + \int_{\mathbf{x}_0}^{\mathbf{x}} \boldsymbol{\gamma}(\mathbf{x}') \cdot \mathrm{d}\mathbf{x}', \quad (3.13)$$

where the gravitational delay effects are separated from the simple plane-wave propagation aspects of the solution. Moreover, let $\hat{\mathbf{k}}$, $\hat{\mathbf{n}}_1$, and $\hat{\mathbf{n}}_2$ form an orthonormal triad in the background Euclidean space such that $\hat{\mathbf{k}} \times \hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_2$. Then $\mathbf{W}_0 = A(\hat{\mathbf{n}}_1 + i\hat{\mathbf{n}}_2)$, where A is a complex amplitude.

In terms of ray propagation, we may interpret these results as implying that a ray starting from $\mathbf{x}_0$ at t_0 will reach $\mathbf{x}$ at t such that

$$t - t_0 = |\mathbf{x} - \mathbf{x}_0| + \delta t_{GE} + \delta t_{GM}, \quad (3.14)$$

where δt_{GE} is the *gravitoelectric time delay* given by

$$\delta t_{GE} = \int_{\mathbf{x}_0}^{\mathbf{x}} [\mathcal{N}(\mathbf{x}') - 1]\, \hat{\mathbf{k}} \cdot \mathrm{d}\mathbf{x}'. \quad (3.15)$$

For $\Phi << 1$, $\mathcal{N} \sim 1 + 2\Phi$ and δt_{GE} reduces to the integral of 2Φ along the straight line from $\mathbf{x}_0$ to $\mathbf{x}$, which is the Shapiro time delay. Moreover, δt_{GM} is the *gravitomagnetic time delay* given by

$$\delta t_{GM} = \int_{\mathbf{x}_0}^{\mathbf{x}} \boldsymbol{\gamma}(\mathbf{x}') \cdot \mathrm{d}\mathbf{x}', \quad (3.16)$$

which vanishes if $\mathbf{x} - \mathbf{x}_0$ is in the same plane as the z-axis. We note that the monopole contribution to Eq. (3.16), modulo an integer multiple of $8\pi\ell$, vanishes in the equatorial plane.

It has recently been shown that the gravitomagnetic time delay caused by rotational motion could make a significant contribution to the gravitational lensing delay of extragalactic sources and should therefore be taken into account in the interpretation of observational data [22, 23, 24]. The time delay around a closed loop would involve a gravitoelectric as well as a gravitomagnetic component. If the loop is traversed in the opposite sense, the gravitoelectric time delay for the stationary field under consideration remains the same but the gravitomagnetic time delay changes sign. In this way, one recovers Eq. (3.6) for the total time delay $t_+ - t_-$ from a different standpoint.

Finally, let us consider the influence of the gravitomagnetic monopole on the polarization of light. The static background Schwarzschild spacetime has no influence on the polarization of test electromagnetic radiation due to its spherical symmetry [19]. We therefore assume that $\Phi \ll 1$ and consider linear perturbations of Minkowski spacetime such that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$,

$$h_{00} = 2\Phi, \quad h_{ij} = 2\Phi\delta_{ij}, \quad h^{0i} = 2A^i_{(\mathrm{g})}. \quad (3.17)$$

Far from the source, $\boldsymbol{\nabla} \cdot \mathbf{A}_{(\mathrm{g})} = 0$ and $\Box\, \mathbf{A}_{(\mathrm{g})} = 0$, so that the trace-reversed perturbations

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \quad (3.18)$$

satisfy $\bar{h}^{\mu\nu}{}_{,\nu} = 0$ and $\Box\, \bar{h}^{\mu\nu} = 0$. It then follows from the analysis in [19, 25] that in a gravitomagnetic field $\mathbf{B}_{(\mathrm{g})}$, the plane of linear polarization rotates at a rate given by [19, 25]

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \mathbf{B}_{(\mathrm{g})} \cdot \hat{\mathbf{k}}\ , \quad (3.19)$$

where ζ is the rotation angle and $\hat{\mathbf{k}}$ is the unit propagation vector of the electromagnetic radiation. This Skrotskii effect [16] is the gravitational analog of the Faraday effect in electrodynamics. Thus for electromagnetic radiation propagating from $\boldsymbol{\rho}_1$ to $\boldsymbol{\rho}_2$, the plane of polarization rotates by the angle $\zeta = \Psi(\boldsymbol{\rho}_1) - \Psi(\boldsymbol{\rho}_2)$ in this linear approximation. In particular, for a ray propagating from a radius $\rho_0 \gg M$ out to infinity, the plane of the polarization rotates by an angle

$$\zeta_0 = \Psi_0 = \frac{\mathbf{J}\cdot\boldsymbol{\rho}_0}{\rho_0^3} + \frac{\mu}{\rho_0} \ . \tag{3.20}$$

This result is consistent with the spherical symmetry of the monopole. Moreover, it follows from Eq. (3.20) that the net angle of rotation of the plane of polarization is zero in the scattering case of radiation propagating from $-\infty$ to ∞.

To go beyond the JWKB limit described above, one must consider the gravitomagnetic phenomenon of helicity-rotation coupling as well; see, for instance, [19, 26] for the case of the Kerr spacetime.

In connection with the spin-rotation coupling, let us consider the motion of a classical spinning test particle in the linearized KTN spacetime given by Eq. (2.2). In the absence of the gravitational source, we may assume that the free test particle is at rest and carries a constant spin vector with its direction fixed in the inertial frame. In the presence of the gravitational source, however, the spin of the particle couples to the spacetime curvature resulting in the Mathisson-Papapetrou force

$$F_\alpha = -\frac{1}{2}R_{\alpha\beta\gamma\delta}\, u^\beta S^{\gamma\delta} \ , \tag{3.21}$$

where u^α is the 4-velocity and $S^{\alpha\beta}$ is the spin tensor of the particle. For a "point" particle with spin, we need to impose the Pirani supplementary condition $S^{\alpha\beta}u_\beta = 0$ for the sake of completeness. The spin vector of the particle is defined by

$$S_\alpha = -\frac{1}{2}(-g)^{1/2}\epsilon_{\alpha\beta\gamma\delta}\, u^\beta S^{\gamma\delta} \ . \tag{3.22}$$

In the linear approximation with $u^\alpha \approx (1,0,0,0)$, $S^{0i} \approx 0$, and $S^{ij} \approx \epsilon^{ijk}S_k$, we find that in Eq. (3.21) the Mathisson-Papapetrou force is $F_0 = 0$ and $F_i = -(\mathbf{B}_{(\mathrm{g})})_{j,i}S^j$. This force may be expressed as the negative gradient of a gravitational spin potential energy $\mathcal{H} = \mathbf{S}\cdot\mathbf{B}_{(\mathrm{g})}$. It turns out that this classical result corresponds in the JWKB approximation to wave-mechanical (quantum) results. Using Eq. (2.7), we see that the energy $\mathcal{H}$ consists of two terms: the familiar gravitational spin-rotation coupling term ($\sim 10^{-28}$eV for a spin-$\frac{1}{2}$ particle in an Earth-based laboratory) and a new term $\mu\mathbf{S}\cdot\boldsymbol{\rho}/\rho^3$ that is due to the coupling of spin with the gravitomagnetic monopole moment of the source.

4. Circular holonomy

Turning now to the parallel transport around closed ϕ-loops, a key result of [27] for the Kerr spacetime involves the relation between the gravitomagnetic temporal structure around a rotating mass and circular equatorial holonomy, namely, the holonomy of the time component $X^t = dt(X)$ of an arbitrary vector field X does not in general vanish for a spacelike circular equatorial orbit around a rotating mass. This temporal holonomy involves a Lorentz boost and the corresponding time dilation indicates the existence of the gravitomagnetic clock effect. The clock effect and the temporal holonomy vanish in the Schwarzschild spacetime. In the case of the Taub-NUT

gravitational field, the corresponding equation of parallel transport around a spacelike circular equatorial orbit implies that

$$\frac{\mathrm{d}X^t}{\mathrm{d}\phi} + \mu X^\theta = 0 \ , \tag{4.1}$$

so the temporal holonomy is in general nonzero for $\ell \neq 0$. This indicates that the gravitomagnetic monopole moment of the source produces a separate temporal structure around the Taub-NUT source as indicated, for instance, by the Skrotskii effect discussed above.

The gravitomagnetic clock effect is essentially absent in this case due to the spherical symmetry of the monopole source, which is a gravitational dyon consisting of a gravitoelectric monopole (represented by the mass M) and a gravitomagnetic monopole (represented by the NUT parameter ℓ). Moreover, timelike circular equatorial geodesic orbits do not exist in the Taub-NUT spacetime. That is, ℓ breaks the degeneracy of the Schwarzschild circular geodesic orbits by lifting them off the equatorial plane in opposite directions. This could be intuitively understood by considering the monopole contribution to the gravitomagnetic Lorentz force per unit mass of the test particle, i.e., $2\mu\boldsymbol{\rho} \times \mathbf{v}/\rho^3$ [28]. To first order in ℓ, however, the orbital frequencies are the same as the Keplerian frequencies in the Schwarzschild spacetime, $\mathrm{d}\phi/\mathrm{d}t = \pm\omega_{(\mathrm{K})}$ with $\omega_{(\mathrm{K})} = (M/r^3)^{1/2}$. This is treated in more detail in the following section.

5. Geodesics in the KNTN spacetime

The motion of a free uncharged test particle in the Kerr-Newman-Taub-NUT spacetime is given by

$$\dot{r} = \pm\frac{1}{\Sigma}\sqrt{R}\ , \qquad R = -\Delta(m^2r^2 + \mathcal{K}) + P^2\ , \quad P = E(r^2 + a^2 + \ell^2) - aL_z\ , \tag{5.1}$$

$$\dot{\theta} = \pm\frac{1}{\Sigma}\sqrt{\Theta}\ , \qquad \Theta = \mathcal{K} - m^2(\ell + a\cos\theta)^2 - \frac{1}{\sin^2\theta}(L_z - E\chi)^2\ , \tag{5.2}$$

$$\dot{\phi} = \frac{1}{\Sigma}\left[\frac{1}{\sin^2\theta}(L_z - E\chi) + a\,\frac{P}{\Delta}\right]\ , \tag{5.3}$$

$$\dot{t} = \frac{1}{\Sigma}\left[\left(r^2 + a^2 + \ell^2\right)\frac{P}{\Delta} + \frac{\chi}{\sin^2\theta}(L_z - E\chi)\right], \tag{5.4}$$

where m, E, L_z, and $\mathcal{K}$ are the four constants of motion for the particle and represent respectively its rest mass, energy, angular momentum about the z-axis, and Carter's constant [29]. Here the dot represents differentiation with respect to the affine parameter λ whose differential is related to the proper time by $\mathrm{d}\tau = m\,\mathrm{d}\lambda$ when $m \neq 0$. For the timelike orbit of a test particle with nonzero rest mass m, the principle of equivalence implies that the orbit is uniquely characterized by the reduced constants $\tilde{E} = E/m$, $\tilde{L}_z = L_z/m$, and $\tilde{\mathcal{K}} = \mathcal{K}/m^2$. A detailed examination of the solutions of these equations is beyond the scope of this article; therefore, we shall only consider certain special orbits here.

Let us first assume that $a = 0$. In this case, we choose $L_z \neq 0$ and $\mathcal{K} = (m^2 - 4E^2)\ell^2 + L_z^2$; one can show from Eq. (5.2) that θ is then fixed and the orbit is confined to a cone with the opening angle θ given by $\cos\theta = -2\ell E/L_z$. It follows that in this case the equations of motion on the cone depend on ℓ only via ℓ^2. As $\ell \to 0$, such an orbit reduces to an equatorial plane orbit; in particular, for

constant r we recover circular equatorial orbits. The reflection of the orbit on the cone about the equatorial plane can be achieved by simply changing the sign of L_z since $\mathrm{d}\phi/\mathrm{d}\tau = L_z/(r^2 + \ell^2)$; that is, the sense of motion is reversed below the equatorial plane. As the gravitomagnetic monopole moment vanishes, the orbits on the cones with $\cos\theta = \pm|2\ell E/L_z|$ degenerate to a single orbit on the equatorial plane.

The absence of timelike circular equatorial geodesics extends to the KTN spacetime. More generally, we can look for geodesic orbits that have constant r and θ coordinates. These would be special circular geodesic orbits around the rotation axis that for $\theta = \pi/2$ coincide with equatorial circular geodesics. A detailed analytic investigation of the geodesic equation shows that for $a \neq 0$ and $\ell \neq 0$ no such (timelike, null, or spacelike) geodesics exist in the KTN spacetime.

Let us next consider timelike spherical geodesics in the KTN spacetime. While such stable orbits exist all the way down to the horizon (for the corotating orbits with $a = M$) in the Kerr case [30, 31], they are bounded away from the horizon of the Kerr-Taub-NUT spacetime, as can be seen by generalizing Fig. 3 of Wilkins [30] and his related calculations. To first order in ℓ, however, the radius of a spherical orbit coincides with that of a Wilkins orbit with the same orbital parameters $(\tilde{E}, \tilde{L}_z, \tilde{\mathcal{K}})$. In this case the orbit follows a spiraling spherical path that stays within latitudes given to first order in ℓ by $\theta_{\min} = \theta_0 + \theta_1$ and $\theta_{\max} = \pi - \theta_0 + \theta_1$. Here $\theta_0 \in (0, \pi/2)$ is the unperturbed latitude satisfying

$$(\tilde{L}_z - a\tilde{E}\sin^2\theta_0)^2 = (\tilde{\mathcal{K}} - a^2\cos^2\theta_0)\sin^2\theta_0 \ . \tag{5.5}$$

If θ_0 is a solution of this equation, then so is $\pi - \theta_0$; therefore, the Wilkins orbits are symmetric in latitude about the equatorial plane. The first order perturbation in latitude θ_1 is obtained from $\Theta = 0$ and can be expressed as

$$\theta_1 = \left(\frac{\ell}{\sin\theta_0}\right) \frac{2\tilde{E}\tilde{L}_z + a(1 - 2\tilde{E}^2)\sin^2\theta_0}{\tilde{\mathcal{K}} + 2a\tilde{E}\tilde{L}_z - a^2 + 2a^2(1 - \tilde{E}^2)\sin^2\theta_0} \ . \tag{5.6}$$

Thus to linear order in the strength of the gravitomagnetic monopole moment, the spherical orbits are shifted up or down and are no longer symmetric in latitude about the equatorial plane.

6. Discussion

Following up on our previous work [10], we have considered some of the main effects associated with the existence of a gravitomagnetic monopole moment. The time coordinate t is periodic in the KNTN spacetime with a period that is simply proportional to the gravitomagnetic monopole moment of the source. The gravitational time delay and the rotation of the plane of polarization of electromagnetic waves studied in this paper further elucidate the special temporal structure in this spacetime. We have also briefly studied the influence of the gravitomagnetic monopole moment on the motion of a spinning particle, and on the spherical orbits in KTN spacetime generalizing some of the previous results of Wilkins [30]. There is no observational evidence at present for the existence of a gravitomagnetic monopole [14, 15]. Our work could in principle be combined with the analysis of astronomical data in order to set upper limits on the possible existence of gravitomagnetic monopoles.

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